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1 Perturbed recombination

1.1 The idea

In overdense (and hotter) regions, there is more recombination[1]. Therefore, the free elecron fraction χ_e in these regions is smaller than in the background. This leads to a lower coupling with CMB photons, hence the gas temperature is lower than it would have been without taking the ionization fraction perturbation $\delta\chi_e$ into account. This in turn decreases the baryonic pressure and increases the formation of structure.

1.2 Notations, conventions.

We use the notation convention which is common in CLASS, CAMB, Recfast and the paper by Anthony Lewis [1].

We work in synchronous gauge. The scale factor is normalized today: $a_0 = 1$. n_H is the total hydrogen (ionized and unionized) number density. n_e is the free electron number density. $\chi_e = n_e/n_H$ is the ionization fraction. $f_{\mathrm{He}} = n_{\mathrm{He}}/n_H$ is the helium-to-hydrogen ratio, it is a *fixed* quantitiy given by nucleosynthesis. Similarly, the helium fraction $Y_{\mathrm{He}} \equiv 4n_{\mathrm{He}}/(n_H + 4n_{\mathrm{He}})$ is fixed.

1.3 Evolution of the free electron fraction perturbation

The ionization fraction obeys

$$\dot{\chi}_e = -a\alpha\chi_e^2 n_H, \tag{1.1}$$

where α is the recombination coefficient,

$$\alpha(T) = F\frac{a_\alpha(T/10^4K)^b}{1 + c(T/10^4K)^b}m^3s^{-1} \tag{1.2}$$

with $F = 1.14, a_\alpha = 4.309\cdot 10^{-19}, b = -0.6166, c = 0.6703, d = 0.53$ (see [1]). Linearizing this equation yields for δ_χ

$$\dot{\delta}_\chi = -a\alpha\chi n_H(\delta_\alpha + \delta_\chi + \delta_{n_H}), \tag{1.3}$$

where δ_{n_H} can be approximated by δ_b and

$$\delta_\alpha = \frac{b + c(T/10^4K)^d(b - d)}{1 + c(T/10^4K)^d}\delta_T. \tag{1.4}$$

[1] After the main recombination epoch, the temperature is well below the ionization energy of the first hydrogen energy state. Therefore, photoionization is very unlikely and the recombination rate depends predominantely on the atomic density and temperature (velocity distribution).

1.4 Evolution of the temperature perturbation

The evolution of fluctuations δ_T in the gas temperature T is derived from the first law of thermodynamics,

$$dQ = \frac{3}{2}dT - T d\log\rho_b, \tag{1.5}$$

where dQ is the heating rate per particle and ρ_b is the baryon density. In the post-recombination era before the formation of galaxies, the only external heating arises from Thomson scattering of the remaining free electrons with CMB photons, resulting in a heating rate per particle

$$\frac{dQ}{dt} = \frac{4\sigma_T\rho_\gamma\chi_e}{m_e c(1+f_{\mathrm{He}}+\chi_e)}(T_\gamma - T). \tag{1.6}$$

Combining these two equations, we obtain

$$\frac{3}{2}\frac{dT}{dt} - T\frac{d\log\rho_b}{dt} = \frac{4\sigma_T\rho_\gamma\chi_e}{m_e c(1+f_{\mathrm{He}}+\chi_e)}(T_\gamma - T) \tag{1.7}$$

In conformal time τ, this leads to

$$\frac{dT}{d\tau} - \frac{2}{3}T\frac{d\log\rho_b}{d\tau} = \frac{8a\sigma_T\rho_\gamma\chi_e}{3m_e c(1+f_{\mathrm{He}}+\chi_e)}(T_\gamma - T) \tag{1.8}$$

This is the evolution equation for the full gas temperature (background + perturbations). To linearize this equation, we use

$$\frac{\dot\rho_b}{\rho_b} = -3\frac{\dot a}{a} + \dot\delta_b \tag{1.9}$$

and

$$\frac{1+\delta_\chi}{1+f+\bar\chi(1+\delta_\chi)} = \frac{1+\delta_\chi}{1+f+\bar\chi} - \frac{\bar\chi\delta_\chi}{(1+f+\bar\chi)^2} \tag{1.10}$$

Then, equation (1.8) can be written at first order as

$$\begin{aligned}&\dot{\bar T}(1+\delta_T) + \bar T\dot\delta_T - \frac{2}{3}\bar T(1+\delta_T)(\dot\delta_b - 3\frac{\dot a}{a}) = \\ &\frac{8a\sigma_T\bar\rho_\gamma(1+\delta_\gamma)\bar\chi}{3m_e c(1+f+\bar\chi)}\left(1+\delta_\chi - \frac{\bar\chi\delta_\chi}{1+f+\bar\chi}\right)\left[\bar T_\gamma(1+\delta_{T_\gamma}) - \bar T(1+\delta_T)\right]\end{aligned} \tag{1.11}$$

At the background level, we have

$$\dot{\bar T} + 2\bar T\frac{\dot a}{a} = \frac{8a\sigma_T\bar\rho_\gamma\bar\chi}{3m_e c(1+f+\bar\chi)}(\bar T_\gamma - \bar T) \tag{1.12}$$

This relation corresponds to equation (7) of Barkana [2] and equation (5) of Lewis [1]. The second term on the left-hand side accounts for adiabatic expansion of the gas, while the right-hand side term captures the effect of the thermal exchange with the CMB. Using the background equation (1.12), equation (1.11) becomes

$$\dot{\delta}_T - \frac{2}{3}\dot{\delta}_b = \frac{8a\sigma_T\bar{\rho}_\gamma\bar{\chi}}{3m_e c(1+f+\bar{\chi})}\left[\left(\frac{\bar{T}_\gamma}{\bar{T}}-1\right)\left(\delta_\chi - \frac{\bar{\chi}\delta_\chi}{1+f+\bar{\chi}}+\delta_\gamma\right)+\frac{\bar{T}_\gamma}{\bar{T}}(\delta_{T_\gamma}-\delta_T)\right]$$

This relation corresponds to equation (6) of Lewis[2] [1] and equation (8) of Barkana[3] [2]. The *background* photon energy density $\bar{\rho}_\gamma$ at time τ can be calculated from thermodynamics relations. In SI units, one has

$$\rho_\gamma = \frac{\pi^2 k_B^4}{15c^3\hbar^3}T_\gamma^4 \equiv a_{\rm rad}a^{-4}T_{\gamma,\rm now}^4 = \rho_\gamma^0 a^{-4}, \tag{1.13}$$

where we have introduced the constant

$$a_{\rm rad} \equiv \frac{\pi^2 k_B^4}{15c^3\hbar^3} = \frac{8\pi^5 k_B^4}{15c^3h^3} = 7.5657\cdot 10^{-16}m^{-1}kgs^{-2}K^{-4}. \tag{1.14}$$

Let us also define

$$t_\gamma^{-1} \equiv \frac{8\sigma_T\rho_\gamma^0}{3m_e c} = 2.7139\cdot 10^{-20}s^{-1} = 8.56\cdot 10^{-13}yr^{-1}, \tag{1.15}$$

so that the evolution equation for the gas temperature perturbation can be written as

$$\dot{\delta}_T = \frac{2}{3}\dot{\delta}_b + \frac{t_\gamma^{-1}\chi a^{-3}}{(1+f+\bar{\chi})}\left[\left(\frac{\bar{T}_\gamma}{\bar{T}}-1\right)\left(\delta_\chi - \frac{\bar{\chi}\delta_\chi}{1+f+\bar{\chi}}+\delta_\gamma\right)+\frac{\bar{T}_\gamma}{\bar{T}}(\delta_{T_\gamma}-\delta_T)\right] \tag{1.16}$$

The temperature perturbation has an impact on the pressure and sound speed of the baryons. This in turn will increase the growth of over-densities. By definition, the speed of sound reads $c_s^2 \equiv \delta p/\delta\rho$. Assuming an ideal, non-relativistic gas, the pressure is given by $p = nT = \rho T/\mu$, where μ is the mean particle mass. Thus,

$$c_s^2\delta_b = \frac{\delta p_b}{\delta\rho_b}\delta_b = \frac{\delta p_b}{\bar{\rho}_b} = \frac{T}{\mu}\left(\delta_b + \delta_T\right) \tag{1.17}$$

We see that the common approximation $c_s^2 \approx dp/d\rho = T/\mu$ is not accurate. In an overdense region($\delta_b > 0$), there is more recombination, thus a lower free electron fraction, i.e. a lower coupling to CMB photons, hence a lower temperature: $\delta_T < 0$. Therefore, the sound speed (and the pressure) is diminished. This has an impact on the growth of large-scale structure. Instead of having

$$\ddot{\delta}_b + 2H\dot{\delta}_b + k^2\frac{c_s^2}{a^2}\delta_b = 4\pi G\sum_i(\delta\rho_i + 3\delta p_i), \tag{1.18}$$

we have

$$\ddot{\delta}_b + 2H\dot{\delta}_b + k^2\frac{c_s^2}{a^2}(\delta_b+\delta_T) = 4\pi G\sum_i(\delta\rho_i + 3\delta p_i). \tag{1.19}$$

[2] We can rewrite $\left(\delta_\chi - \frac{\bar{\chi}\delta_\chi}{1+f+\bar{\chi}}\right)$ as $\frac{(1+f)\delta_\chi}{1+f+\bar{\chi}} = \frac{\delta_\chi}{1+n_H\bar{\chi}/(n_H+n_{\rm He})}$.

[3] In this paper they neglect δ_χ

1.5 Summary

We have a system of two coupled first-order differential equations:

$$\dot{\delta}_\chi = -a\alpha\chi n_H(\delta_\alpha + \delta_\chi + \delta_b)$$
$$\dot{\delta}_T = \frac{2}{3}\dot{\delta}_b + \frac{t_\gamma^{-1}\chi a^{-3}}{(1+f+\bar{\chi})}\left[\left(\frac{\bar{T}_\gamma}{\bar{T}} - 1\right)\left(\delta_\chi - \frac{\bar{\chi}\delta_\chi}{1+f+\bar{\chi}} + \delta_\gamma\right) + \frac{\bar{T}_\gamma}{\bar{T}}(\delta_{T_\gamma} - \delta_T)\right]$$

With the initial conditions $\delta_\chi = 0$ and $\delta_T = \frac{1}{3}\delta_b$. Of course, we have $\delta_{T_\gamma} = \frac{1}{4}\delta_\gamma$.

Remark concerning Runge-Kutta evolver: We have to start the system well before δ_T differs from δ_B, so that our initial conditions remain valid. At such times, δ_{T_γ} oscillates quickly due to baryonic accoustic oscillations. In the evolution equation for δ_T, there is a strong restoring term $(\delta_T - \delta_{T_\gamma})$. Therefore, the time derivative will be very important, and the Runge-Kutta evolver will set a very small time scale, which will slow down the code. Consequently, one should use ndf15 instead.

References

[1] A. Lewis, "Linear effects of perturbed recombination," *Phys.Rev.* **D76** (2007) 063001, `arXiv:0707.2727 [astro-ph]`.

[2] S. Naoz and R. Barkana, "Growth of linear perturbations before the era of the first galaxies," *Mon.Not.Roy.Astron.Soc.* **362** (2005) 1047–1053, `arXiv:astro-ph/0503196 [astro-ph]`.