PPF formalism in the synchronous and Newtonian gauges

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1 PPF formalism

In the PPF formalism we define the parameter Γ which is directly related to the DE energy density perturbation in the DE rest frame:

$$\Gamma = -\frac{4\pi G a^2}{k^2 c_K}\delta\varrho_{\rm fld}^{\rm rest} = -\frac{3}{2}\frac{a^2}{k^2 c_K}\delta\rho_{\rm fld}^{\rm rest}. \tag{1}$$

Here c_K is in the notation from Hu, but in CLASS this is just s_2^2 available in `ppw->s_l[2]`. We use ϱ to denote physical energy densities and ρ to denote CLASS rescaled energy densities. In the minimal implementation of Ref. [1] we set $f_\zeta = 0$ and $c_\Gamma = 0.4 c_{s,\rm fld}$ where $c_{s,\rm fld}$ is the sound speed in the fluid rest-frame. The evolution equation for Γ is

$$\Gamma' = \frac{a'}{a}\left[\frac{S}{1+\frac{c_\Gamma^2 k^2}{a^2H^2}} - \Gamma\left(1+\frac{c_\Gamma^2 k^2}{a^2H^2}\right)\right]. \tag{2}$$

Note that our primes denote derivation w.r.t. conformal time like in CLASS. The source term S is given by

$$S = \frac{a'}{a}\frac{4\pi G}{H^2}\varrho_{\rm fld}(1+w_{\rm fld})\frac{\theta_T^{(N)}}{k^2} = \frac{3}{2}\left(\frac{a'}{a}\right)^{-1}\frac{a^2}{k^2}\rho_{\rm fld}(1+w_{\rm fld})\theta_T^{(N)}. \tag{3}$$

In this equation subscript T denotes all species except dark energy, and the superscript (N) refers to Newtonian gauge. We will now recover the DE contributions to $\delta\rho$ and $(\rho+p)\theta$. From any gauge we can transform into a frame where the DE is at rest by the following transformation:

$$\delta\rho_{\rm fld}^{\rm rest} = \delta\rho_{\rm fld} + 3\frac{a'}{a}\rho_{\rm fld}(1+w_{\rm fld})\frac{\theta_{\rm fld}}{k^2}. \tag{4}$$

From this equation we can compute $\delta\rho_{\rm fld}$ in any gauge if $\theta_{\rm fld}$ and Γ are known. Starting from Eq. 40 in [2], we get

$$\left(\rho_{\rm fld}+p_{\rm fld}\right)\left(\theta_{\rm fld}-\theta_T\right) = -\frac{2}{3}\frac{k^2}{a^2}\frac{a'}{a}\frac{S-\Gamma-\left(\frac{a'}{a}\right)^{-1}\Gamma'}{1+\frac{9}{2}\frac{a^2}{k^2c_K}(\rho_T+p_T)}. \tag{5}$$

Since the left hand side is a relative velocity, it is gauge invariant. We can then use this expression to compute $(\rho_{\rm fld}+p_{\rm fld})\theta_{\rm fld}$ in any gauge. These equations can now be implemented directly into CLASS in the Newtonian gauge. For the synchronous gauge, one essential equation is missing: To compute S we must know $\theta_T^{(N)}$. From the gauge transformation we have

$$\theta_T^{(N)} = \theta_T^{(S)} + k^2\alpha, \tag{6}$$

where $\alpha \equiv \frac{h'+6\eta'}{2k^2}$.

1.1 Flat universe:

From the first two Einstein equations we find

$$\begin{aligned}
\alpha &= \left(\frac{a'}{a}\right)^{-1}\left[\eta + \frac{3}{2}\frac{a^2}{k^2}\delta\rho\right] + \frac{9}{2}\frac{a^2}{k^4}(\rho+p)\theta & (7)\\
&= \left(\frac{a'}{a}\right)^{-1}\left[\eta + \frac{3}{2}\frac{a^2}{k^2}\delta\rho_T\right] + \frac{9}{2}\frac{a^2}{k^4}(\rho_T+p_T)\theta_T + & (8)\\
&+ \left(\frac{a'}{a}\right)^{-1}\frac{3}{2}\frac{a^2}{k^2}\delta\rho_{\rm fld} + \frac{9}{2}\frac{a^2}{k^4}(\rho_{\rm fld}+p_{\rm fld})\theta_{\rm fld} & (9)\\
&= \left(\frac{a'}{a}\right)^{-1}\left[\eta + \frac{3}{2}\frac{a^2}{k^2}\delta\rho_T\right] + \frac{9}{2}\frac{a^2}{k^4}(\rho_T+p_T)\theta_T + & (10)\\
&+ \left(\frac{a'}{a}\right)^{-1}\frac{3}{2}\frac{a^2}{k^2}\left[\delta\rho_{\rm fld} + 3\frac{a'}{a}(\rho_{\rm fld}+p_{\rm fld})\frac{\theta_{\rm fld}}{k^2}\right] & (11)\\
&= \left(\frac{a'}{a}\right)^{-1}\left[\eta + \frac{3}{2}\frac{a^2}{k^2}\delta\rho_T\right] + \frac{9}{2}\frac{a^2}{k^4}(\rho+p)\theta_T - \left(\frac{a'}{a}\right)^{-1}\Gamma & (12)
\end{aligned}$$

The last equation follows from the definition of Γ.

1.2 Non-flat universe:

We combine the two first Einstein equations,

$$\begin{aligned}
h' &= 2\left(\frac{a'}{a}\right)^{-1}\left[k^2 c_K\eta + \frac{3}{2}a^2\delta\rho\right], & (13)\\
\eta' &= \frac{1}{k^2 c_K}\left[\frac{3}{2}a^2(\rho+p)\theta + \frac{1}{2}Kh'\right], & (14)
\end{aligned}$$

to form

$$\begin{aligned}
6\eta' + h' &= \frac{1}{k^2 c_K}\left[9a^2(\rho+p)\theta + 3Kh'\right] + h', & (15)\\
&= \frac{1}{k^2 c_K}\left[9a^2(\rho+p)\theta + k^2 h'\right]. & (16)
\end{aligned}$$

We can now compute α:

$$
\begin{aligned}
\alpha &= \frac{1}{2k^4 c_K}\left[9a^2(\rho+p)\theta + 2k^2\left(\frac{a'}{a}\right)^{-1}\left(k^2 c_K \eta + \frac{3}{2}a^2\delta\rho\right)\right], && (17)\\
&= \frac{1}{k^2 c_K}\left[\left(\frac{a'}{a}\right)^{-1} k^2 c_K \eta + \frac{9}{2}\frac{a^2}{k^2}(\rho+p)\theta + \left(\frac{a'}{a}\right)^{-1}\left(\frac{3}{2}a^2\delta\rho\right),\right], && (18)\\
&= \left(\frac{a'}{a}\right)^{-1}\left[\eta + \frac{3}{2}\frac{a^2}{k^2 c_K}\left(\delta\rho + 3\frac{a'}{a}(\rho+p)\theta\right)\right], && (19)\\
&= \left(\frac{a'}{a}\right)^{-1}\left[\eta + \frac{3}{2}\frac{a^2}{k^2 c_K}\left(\delta\rho_T + 3\frac{a'}{a}(\rho+p)\theta_T\right) - \Lambda\right]. && (20)
\end{aligned}
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References

[1] Wenjuan Fang, Wayne Hu, and Antony Lewis. Crossing the Phantom Divide with Parameterized Post-Friedmann Dark Energy. *Phys. Rev.*, D78:087303, 2008.

[2] Wayne Hu. Parametrized Post-Friedmann Signatures of Acceleration in the CMB. *Phys. Rev.*, D77:103524, 2008.